

Quark current correlators and precise charm and bottom masses

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LoopFest IX



- 1 Motivation
- 2 Theoretical moments
- 3 Experimental moments
- 4 Charm and bottom masses

Motivation

Why precise quark masses?

- Fundamental Standard Model parameters
- Flavour physics, e.g. semileptonic B decays
- $H \rightarrow b\bar{b}$
- “By-products:” $\alpha_s, \sigma(e^+e^- \rightarrow \text{hadrons})$

Quark mass determination

from experiment

Charm and bottom masses from

theory

$$\Pi(q^2) \sim \text{diagram}$$

experiment

$$R(s) = \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)}$$

using the dispersion relation

$$\Pi(q^2) = \frac{q^2}{12\pi^2} \int_0^\infty ds \frac{R(s)}{s(s-q^2)}$$

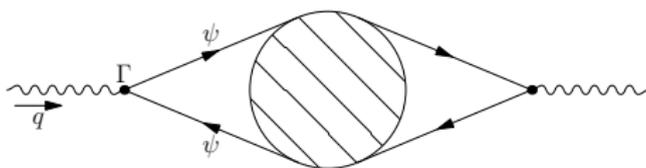
expand around $q^2 = 0$:

$$\frac{3Q^2}{16\pi^2} \sum_n C_n \left(\frac{q^2}{4m^2} \right)^n = \sum_n (q^2)^n \frac{1}{12\pi^2} \underbrace{\int ds \frac{R(s)}{s^{n+1}}}_{\mathcal{M}_n^{\text{exp}}}$$

$$m = \frac{1}{2} \left(\frac{9Q^2}{4} \frac{C_n}{\mathcal{M}_n^{\text{exp}}} \right)^{\frac{1}{2n}}$$

Quark current correlators

$$\Pi(q^2) \propto i \int dx e^{iqx} \langle 0 | T j(x) j(0) | 0 \rangle$$



- $j = \bar{\psi} \Gamma \psi$ with $\psi \in \{c, b\}$ and $\Gamma \in \{\mathbf{1}, \gamma_\mu, \gamma_5, \gamma_5 \gamma_\mu\}$

Theoretical moments

[Källén, Sabry '55; Chetyrkin, Kühn, Steinhauser '96; Chetyrkin, Kühn, Steinhauser '97; Kühn, Steinhauser, Sturm '06; Boughezal, Czakon, Schutzmeier '06; Sturm '08; AM, Maierhöfer, Marquard '08; AM, Maierhöfer, Marquard, Smirnov '09]

$$\Pi(q^2) = \frac{3Q^2}{16\pi^2} \sum_n C_n \left(\frac{q^2}{4m^2} \right)^n$$

Perturbative expansion:

$$C_n = C_n^{(0)} + C_n^{(1)} \frac{\alpha_s}{\pi} + C_n^{(2)} \left(\frac{\alpha_s}{\pi} \right)^2 + C_n^{(3)} \left(\frac{\alpha_s}{\pi} \right)^3 + \dots$$

$$C_n^{(0)} \hat{=} \text{circle} \quad \dots \quad C_n^{(3)} \hat{=} \text{diagrams with 8 dots}$$

Larger $n \Rightarrow$ more “dots” (higher powers of propagators)
more irreducible scalar products
higher number of scalar integrals

$C_3^{(3)}$: $\sim 4\,000\,000$ scalar integrals, up to 12 dots, 8 scalar products

Theoretical moments

Calculation

- 1 Reduce to scalar integrals

$$\int dk_1 \int dk_2 \cdots \frac{1}{D_1^{a_1} D_2^{a_2} \cdots}$$

- 2 Generate integration by parts (IBP) relations [Chetyrkin, Tkachov '81]

$$\int dk_1 \int dk_2 \cdots \frac{\partial}{\partial k_i} k_j \frac{1}{D_1^{a_1} D_2^{a_2} \cdots} = 0$$

- 3 Solve resulting linear system of relations between scalar integrals

[Laporta '00, Gauß '09]

- 4 Insert master integrals (all known)

[Chetyrkin et al. '05-'06; Schröder, Steinhauser '05-'06; Laporta '02; Kniehl et al. '06]

Higher moments

$$\Pi(q^2) = \frac{3Q^2}{16\pi^2} \sum_n C_n \left(\frac{q^2}{4m^2} \right)^n$$

Idea: $\Pi(q^2)$ for arbitrary $q^2 \xrightarrow{\text{Taylor}} C_n$

Approximate $\Pi(q^2)$ using known expansions:

- low energies (moments C_n)
- threshold [Hoang, ... '00, Penin, Pivarov '99, Beneke, Signer, Smirnov '99, Pineda, Signer '07]
- high energies

[Gorishnii, Kataev, Larin '91; Baikov, Chetyrkin, Harlander, Kühn, Steinhauser '97-'09]

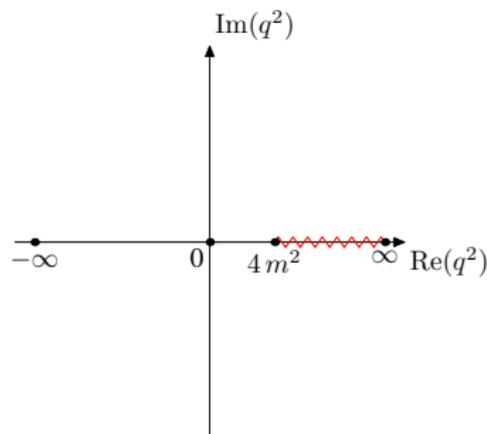
Use Padé approximation:

$$p_{n,m}(x) = \frac{a_0 + a_1x^1 + \dots + a_nx^n}{1 + b_1x^1 + \dots + b_mx^m}$$

Padé approximation

[Broadhurst, Fleischer, Tarasov '93; Baikov, Broadhurst '95; Chetyrkin, Kühn, Steinhauser '96; Hoang, Mateu, Zerbarjad '08; Masjuan, Peris '08; Kiyo, AM, Maierhöfer, Marquard '09]

Problem: $\Pi(q^2)$ has a *branch cut* for $q^2 > 4m^2$



Logarithms in threshold and high energy expansions:

$$\begin{aligned} \Pi^{(3),v}(q^2) \xrightarrow{q^2 \rightarrow -\infty} &= -6.172 - 0.06988 \log\left(-\frac{q^2}{m^2}\right) + 0.1211 \log^2\left(-\frac{q^2}{m^2}\right) \\ &\quad - 0.03665 \log^3\left(-\frac{q^2}{m^2}\right) + \dots \end{aligned}$$

Padé approximation

Solution:

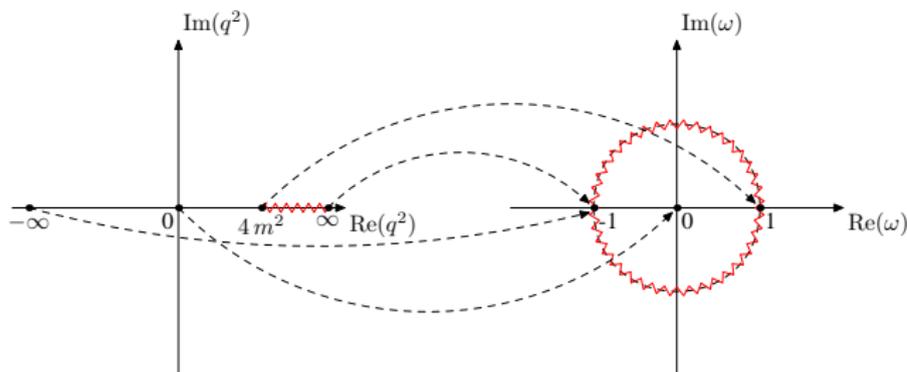
- 1 Subtraction of an appropriate function (not unique):

$$\Pi(q^2) = \Pi_{\text{reg}}(q^2) + \Pi_{\text{log}}(q^2)$$

⇒ construction of many different approximants
(error estimate)

- 2 Transformation:

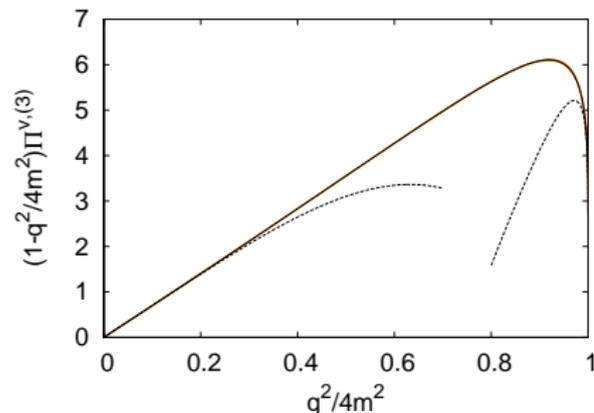
$$\frac{q^2}{4m^2} = \frac{4\omega}{(1 + \omega)^2}$$



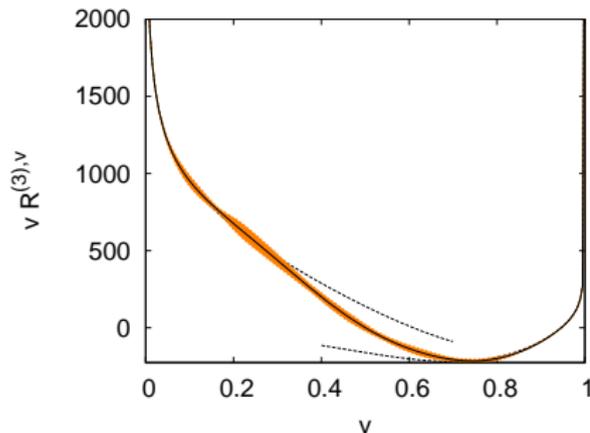
Padé approximation

Vector correlator

below threshold



above threshold
(absorptive part)

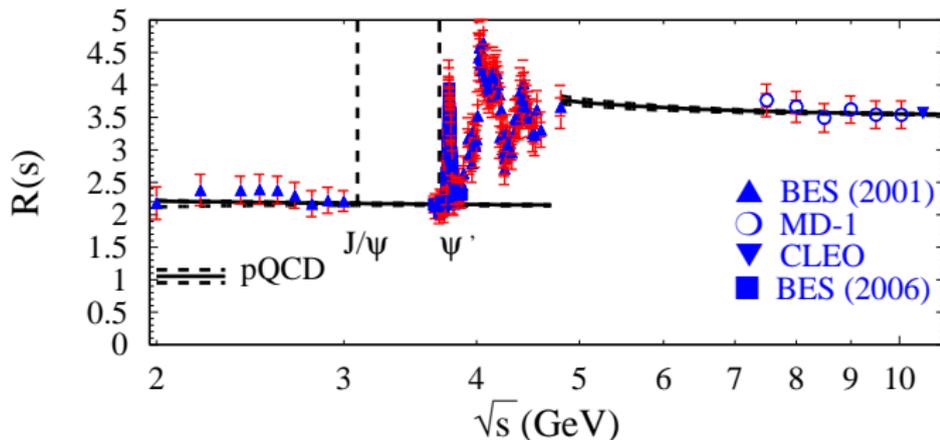


- Approximation very good below threshold
- Good estimates for four loop moments $C_n^{(3)}$ up to $n \approx 10$
- By-product: $\sigma(e^+e^- \rightarrow \text{hadrons})$

Experimental moments

Charm

$$\mathcal{M}_n^{\text{exp}} = \int ds \frac{R(s)}{s^{n+1}}$$

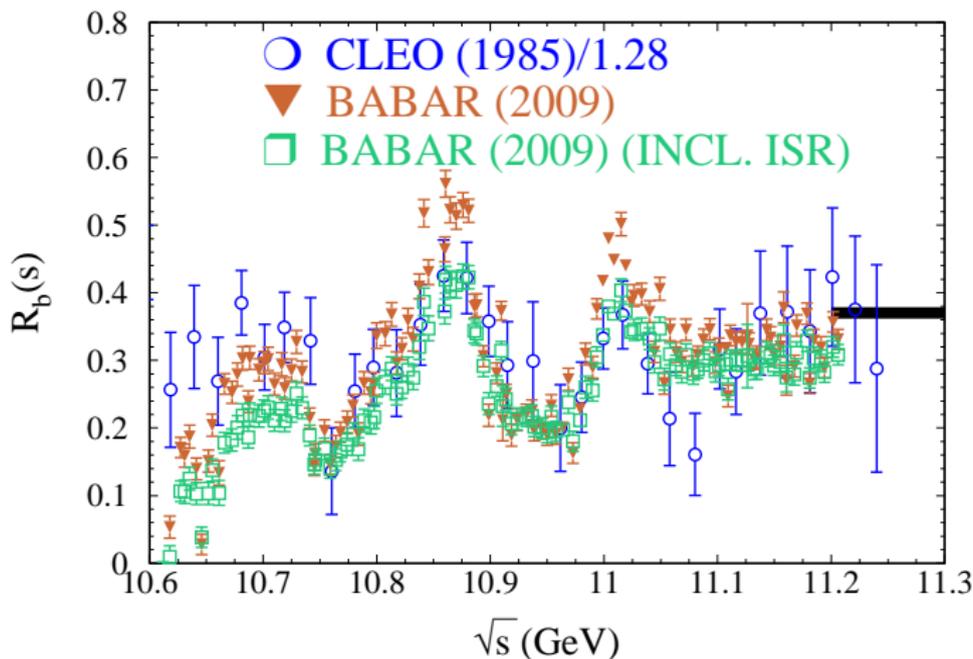


Contributions from

- narrow resonances: $J/\psi, \psi'$ (PDG)
- threshold: $2m_D \leq E < 4.8$ GeV (BES)
- continuum: $E \geq 4.8$ GeV (theory)

Experimental moments

Bottom



Contributions from

- narrow resonances: $\Upsilon(1S) - \Upsilon(4S)$ (PDG)
- threshold: $10.6 \text{ GeV} \leq E < 11.2 \text{ GeV}$ (BaBar)
- continuum: $E \geq 11.2 \text{ GeV}$ (theory)

Charm mass

[Chetyrkin, Kühn, AM, Maierhöfer, Marquard, Steinhauser, Sturm '09]

n	$m_c(3 \text{ GeV})$	exp	α_s	μ	np	total
1	986	9	9	2	1	13
2	976	6	14	5	0	16
3	978	5	15	7	2	17
4	1004	3	9	31	7	33

- Good agreement between different moments
- Smallest error for $n = 1$

$$m_c(3 \text{ GeV}) = 986(13) \text{ MeV}$$

Bottom mass

[Chetyrkin, Kühn, AM, Maierhöfer, Marquard, Steinhauser, Sturm '09]

n	$m_b(10 \text{ GeV})$	exp	α_s	μ	total	$m_b(m_b)$
1	3597	14	7	2	16	4151
2	3610	10	12	3	16	4163
3	3619	8	14	6	18	4172
4	3631	6	15	20	26	4183

- Agreement between different moment rise with increasing n ?
- Comparable uncertainties for $n = 1, 2, 3$
- Small theory error and continuum contribution for $n = 2$

$$m_b(10 \text{ GeV}) = 3610(16) \text{ MeV}$$

$$m_b(m_b) = 4163(16) \text{ MeV}$$

Conclusion

Precise *charm* and *bottom quark masses* from moments of

- *R-ratio* (BES & BaBar)
- *quark current correlators* (4-loop perturbative QCD+Padé)

$$m_c(3 \text{ GeV}) = 986(13) \text{ MeV}$$

$$m_b(m_b) = 3610(16) \text{ MeV}$$
$$m_b(10 \text{ GeV}) = 4163(16) \text{ MeV}$$

Lattice determination

[HPQCD, '09-'10]

Use moments from lattice instead of experimental data:

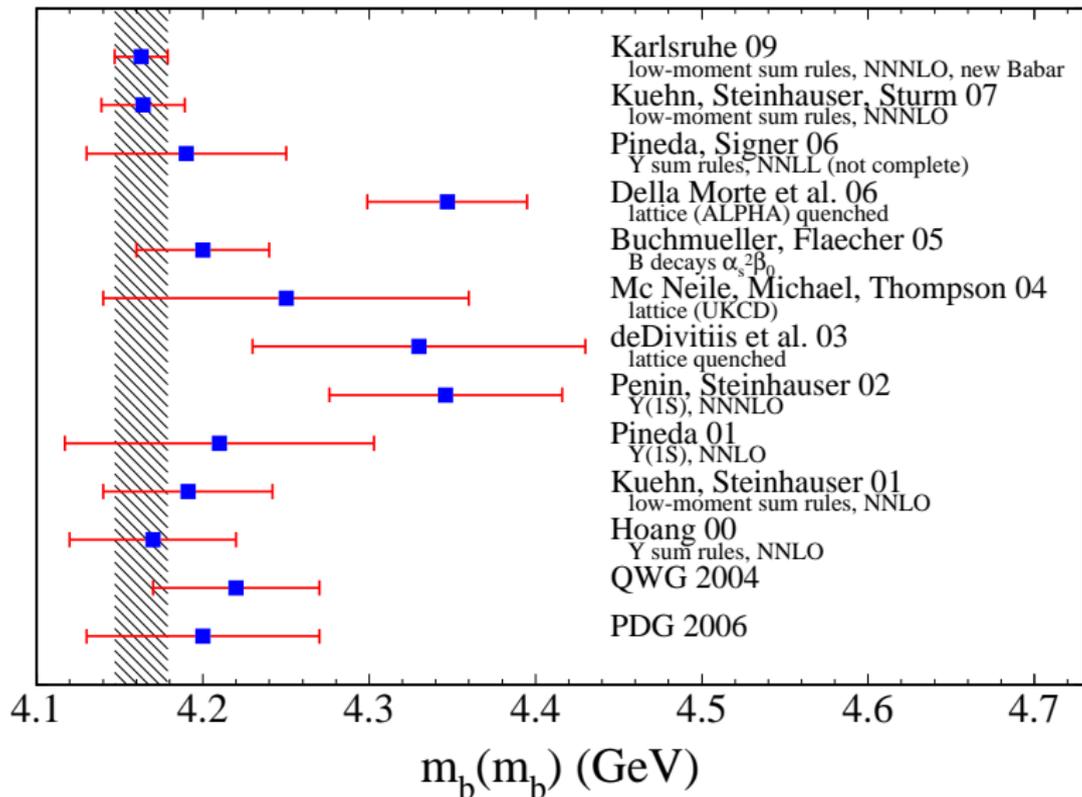
$$G(t) = a^6 \sum_{\mathbf{x}} \langle 0 | j(\mathbf{x}, t) j(0, 0) | 0 \rangle$$
$$G_n = \sum_t (t/a)^n G(t)$$

- Works for all correlators (best for pseudoscalar)
- m_c , m_b , α_s from fit of lattice to perturbative moments

$$m_c(3 \text{ GeV}) = 986(6) \text{ MeV}$$

$$m_b(m_b) = 4164(25) \text{ MeV}$$

$$\alpha_s(M_Z) = 0.1183(7)$$



Selected references

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